The Wilson Loop in Classical Lattice Gauge Theory and the Thermal Width of Heavy Quarkonium

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Project description

Abstract: We present an estimate for the imaginary part of the recently introduced real-time static potential. It can be extracted from the time evolution of the Wilson loop in classical lattice gauge theory. The real-time static potential determines, through a Schrödinger-type equation and a subsequent Fourier-transform of its solution, the spectral function of heavy quarkonium in finite-temperature QCD. We also compare the results of the classical simulations with those of Hard Thermal Loop improved simulations, as well as with analytic expectations based on resummed perturbation theory.

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Real-time static potential

Heavy quarkonium spectral function

The heavy quarkonium spectral function in the vector channel, $\rho(\omega)$, can be obtained through

$$\rho(\omega) = \frac{1}{2}\left(1 - e^{-\frac{\omega}{T}}\right)\int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t} C_>(t, \mathbf{0}) \qquad (1)$$

from the mesonic correlator

$$C_>(t,\mathbf{r}) \equiv \int \mathrm{d}^3\mathbf{x} \left\langle \hat{\bar{\psi}}\left(t, \mathbf{x}+\frac{\mathbf{r}}{2}\right)\gamma^\mu W\, \hat{\psi}\left(t, \mathbf{x}-\frac{\mathbf{r}}{2}\right) \hat{\bar{\psi}}(0,\mathbf{0})\gamma_\mu \hat{\psi}(0,\mathbf{0}) \right\rangle , \qquad (2)$$

where a point-splitting has been introduced to facilitate a perturbative treatment. W denotes a Wilson line connecting the adjacent operators along a straight path. The dilepton production rate from $q\bar{q}$-annihilation at temperature T is proportional to the spectral function.

Definition of a real-time static potential

Focusing on infinitely heavy quarks the correlator can be obtained, up to normalization and a trivial phase factor, from the analytic continuation of a euclidean Wilson loop [1],

$$C_>(t,\mathbf{r}) \propto C_E(it,\mathbf{r}), \quad C_E(\tau,\mathbf{r}) = \frac{1}{N_c}\mathrm{Tr}\,\langle W(0,\mathbf{r};\tau,\mathbf{r}) W(\tau,\mathbf{r};\tau,0) W(\tau,0;0,0) W(0,0;0,\mathbf{r})\rangle \ .$$

At $t \neq 0$ we can write the time evolution in the form of a Schrödinger equation,

$$[i\partial_t - V_>(t,r)]\, C_>(t,\mathbf{r}) = 0 \ , \quad r \equiv |\mathbf{r}| \ , \qquad (3)$$

which defines the object $V_>$ which we refer to as the *real-time static potential.*

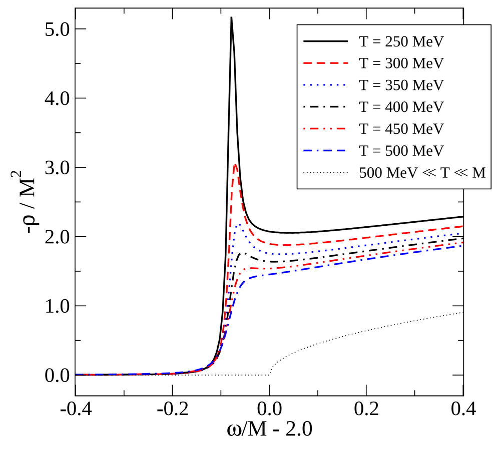

FIGURE 1: The resummed perturbative bottomonium contribution to the spectral function in the non-relativistic regime [2].

Results from perturbation theory

An analytic determination of the Wilson loop using HTL-resummed perturbation theory yields the following result in the large-time limit [1]: $V_>(\infty, r) =$

$$= -\frac{g^2 C_F}{4\pi}\left[m_D + \frac{\exp(-m_D r)}{r}\right] - \frac{ig^2 T C_F}{4\pi}\phi(m_D r) \ ,$$

$$\text{with} \quad \phi(x) = 2\int_0^\infty \frac{\mathrm{d}z\, z}{(z^2+1)^2}\left[1 - \frac{\sin(zx)}{zx}\right] . \qquad (4)$$

The real part corresponds to the standard Debye-screened potential of a static quark pair. The Debye mass is denoted by m_D. The imaginary part of the potential controls the damping of the correlator in eq. (3).

Physical role of the imaginary part: The spectral function in fig. 1 is obtained by inserting the potential into a Schrödinger equation [like eq. (3) but supplemented by the usual mass terms and spatial derivatives], and employing subsequently eq. (1). The imaginary part of the potential introduces a thermal width to the tip of the quarkonium peak.

Goal of the simulation

The goal of the simulation is to assess the extent of non-perturbative corrections to the analytic $V_>(t,r)$. As usual, a direct analytic continuation from numerical data for $C_E(\tau,\mathbf{r})$ is not feasible. However, it turns out that the imaginary part of $V_>$ is formally classical [1], and can hence be probed non-perturbatively with classical lattice gauge theory simulations, of the type originally introduced by Grigoriev and Rubakov [4].

Classical lattice gauge theory simulations

Lattice setup

The framework of classical lattice gauge theory simulations [4] is quite similar to the Kogut-Susskind Hamiltonian approach [5]:

- The fields are discretized using a 3-dimensional spatial lattice. The time coordinate remains continuous.
- Besides the spatial links U_i, corresponding to the discretized colour-magnetic fields, an electric field E_i is defined via the relation $\dot{U}_i(x) = iE_i(x)U_i(x)$, where $x \equiv (t, \mathbf{x})$ and $\dot{U} \equiv \partial U/\partial t$.
- A temporal gauge is chosen. The space of physical states is constrained to gauge field configurations satisfying the discretized Gauss law,

$$G(x) \equiv \sum_i \left[E_i(x) - P_{-i}(x)E_i(x-\hat{i})\right] - j^0(x) \equiv 0 \ , \qquad (5)$$

with j^μ denoting a possible colour current, and P_i the adjoint parallel transporter, $P_i\phi(x+\hat{i}) = U_i(x)\phi(x+\hat{i})U_i^\dagger(x)$.

Classical Yang-Mills fields on the lattice

The classical approximation for Yang-Mills fields at finite temperature follows by supplementing the phase space just introduced with a canonical time evolution and an average over initial conditions with a thermal weight. The weight corresponds to the one in the classical partition function

$$Z = \int \mathcal{D}U_i\, \mathcal{D}E_i\, \delta(G) e^{-\beta H} \ , \quad H = \frac{1}{N_c}\sum_x \left[\sum_{i<j} \mathrm{Re\,Tr}\,(1 - U_{ij}) + \frac{1}{2}\mathrm{Tr}\,(E_i^2)\right] , \qquad (6)$$

where U_{ij} is the plaquette. The classical equations of motion for the discretized system can be obtained by invoking the Hamiltonian principle $\delta S = 0$, and read $[\mathrm{Tr}\,(T^a T^b) = \delta^{ab}/2]$ [6]:

$$\dot{U}_i(x) = iE_i(x)U_i(x) \ , \quad E_i = \sum_a E_i^a T^a \ , \quad \dot{E}_i^a(x) = -2\,\mathrm{Im\,Tr}\,[T^a \sum_{|j|\neq i} U_{ij}(x)] \ . \qquad (7)$$

Hard Thermal Loop effective theory

A more thorough treatment of the long-range dynamics of hot QCD is possible by using the so-called Hard Thermal Loop (HTL) effective theory [7], which is obtained by integrating out the "hard modes" (with momenta of the order of the temperature) from the system, in order to construct an effective theory for the soft modes. In order to keep the effective theory local, certain on-shell particle degrees of freedom need, however, to be added to the effective Hamiltonian [8]. Once this system is discretized and the classical limit is taken, the properties of the hard modes change, and the associated matching coefficient, denoted by m_D^2, needs to be tuned correspondingly [9]. In the following we denote the new on-shell particle modes by $W(x,v)$. In a numerical implementation the main changes with respect to the purely classical theory are as follows:

1. The Hamiltonian obtains an additional part,

$$\delta H = \frac{1}{N_c}\sum_x \left[\int \frac{\mathrm{d}\Omega_v}{4\pi}\frac{1}{2}(am_D)^2 \mathrm{Tr}\,(W^2)\right] , \qquad (8)$$

where $W \equiv T^a W^a(x,v)$ describes the charge density of the on-shell modes at x moving in the direction $v = (1, \mathbf{v})$.

2. The velocities $\mathbf{v}$ need to be discretised. This could be done, for instance, with spherical harmonics [10] or with platonian solids [11]. Choosing the latter approach, we replace $\int \mathrm{d}\Omega_v/4\pi f(v) \to 1/N_p \sum_{n=1}^{N_p} f(v_n)$. The equation of motion of the gauge fields then gets the source term

$$j^\mu(x) = (am_D)^2 \frac{1}{N_p}\sum_{n=1}^{N_p} v_n^\mu W_n(x) \ , \quad W_n(x) \equiv W(x, v_n) \ . \qquad (9)$$

3. Finally, the new fields also evolve in time, according to

$$\dot{W}_n(x) = v_n^i \left(\bar{E}_i(x) - \frac{1}{2}\left[P_i W_n(x+\hat{i}) - P_{-i}W_n(x-\hat{i})\right]\right) , \qquad (10)$$

where $\bar{E}_i(x) \equiv [E_i(x) + P_{-i}E_i(x-\hat{i})]/2$.

Imaginary part of the real-time static potential from Wilson loop dynamics

The Wilson loop as a real-time observable

To obtain the imaginary part of the real-time static potential a rectangular Wilson loop of spatial extent $r = |\mathbf{r}|$ and temporal extent t was measured using classical or HTL-improved lattice simulations. The average over a statistical ensemble of initial configurations, as well as over lattice sites and loop orientations, is denoted by $C_{\mathrm{cl}}(t,r)$.

Generation of the ensemble

The set of initial configurations respecting the Gauss law and distributed according to the statistical weight in eq. (6) was created using the following algorithm:

1. Pre-generate the spatial gauge links U_i with a 3d Monte Carlo simulation.
2. Generate the electric fields from a gaussian distribution [cf. eq. (6)].
3. Project onto the space of physical configurations, satisfying the Gauss law [10].
4. Evolve the fields using the EOM, and repeat from step 2, until the fields have thermalized.

For the HTL-improved simulations the ensemble was generated using a similar procedure.

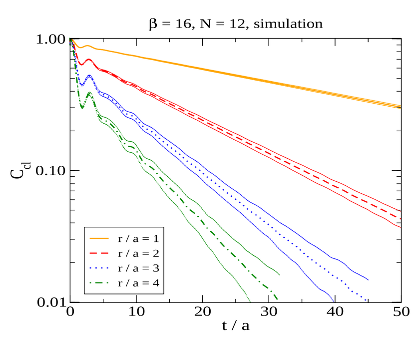

FIGURE 2: The correlator $C_{\mathrm{cl}}(t,r)$ [3]. The corresponding potential is shown in fig. 3.

Calculating the imaginary part of the potential

The real-time static potential can be calculated from eq. (3):

$$V_{\mathrm{cl}}(t,r) \equiv \frac{i\partial_t C_{\mathrm{cl}}(t,r)}{C_{\mathrm{cl}}(t,r)} \ , \quad C_{\mathrm{cl}}(t,r) \equiv \frac{1}{N_c}\mathrm{Tr}\,\left\langle W_r^\dagger(t) W_r(0)\right\rangle \qquad (11)$$

with $W_r(t)$ denoting a spatial Wilson line of length r. Timelike Wilson lines have disappeared due to the temporal gauge. The result obtained for $V_{\mathrm{cl}}(t,r)$ is purely imaginary.

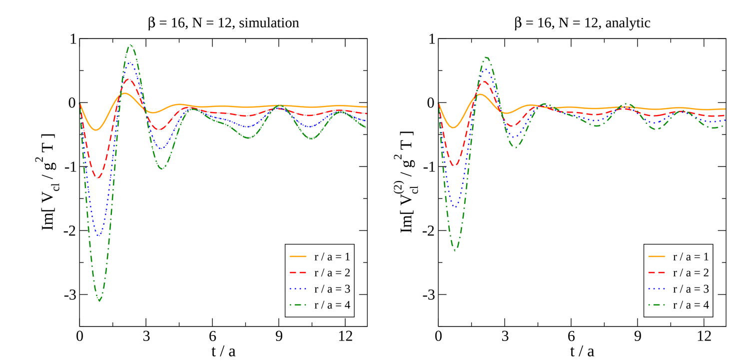

FIGURE 3: The imaginary part of the real-time static potential from the classical simulation and from perturbation theory [3].

	β_3	N	am_D	confs	$r = 1a$	$r = 2a$	$r = 3a$	$r = 4a$
Simulation	16.0	12	0.0	200	-0.060(2)	-0.156(8)	-0.246(26)	-0.319(56)
	16.0	16	0.0	160	-0.059(2)	-0.155(8)	-0.245(22)	-0.326(48)
	16.0	12	0.211	200	-0.059(2)	-0.147(7)	-0.229(23)	-0.297(51)
	16.0	12	0.350	182	-0.030(2)	-0.064(5)	-0.096(12)	-0.118(21)
	13.5	12	0.250	142	-0.071(2)	-0.174(10)	-0.270(33)	-0.341(97)
Analytic	16.0	∞	0.0	-	-0.0601	-0.1145	-0.1507	-0.1737

TABLE 1: Overview of the results in the large-time limit [3]. The results from the classical and HTL-improved simulations agree within error bars for $am_D < 0.25$ (at $\beta = 16$).

Comparison with analytic expectations

As shown in fig. 3, the results obtained from perturbation theory and from the classical numerical simulations are remarkably similar. At the same time, some amplification of the imaginary part through the inclusion of non-perturbative (and higher-order perturbative) effects is visible in the simulation result.

The difference between the two results becomes more pronounced at later times. In particular, in the large-time limit, a difference between the perturbative and the numerical results of up to $\sim 100\%$ can be observed (at $\beta = 16$), cf. table 1.

Conclusion

The results from the real-time lattice simulations confirm the existence of an imaginary part in the real-time static potential, indicated already by leading-order Hard Loop resummed perturbation theory. In fact, non-perturbative and higher order perturbative corrections ***amplify*** the imaginary part by up to $\sim 100\%$. The imaginary part widens (and lowers) the quarkonium peaks in fig. 1, although the qualitative structures remain unchanged. As a side remark, we note that the existence of an imaginary part also leads to strong damping in the solution of the Schrödinger equation, thus significantly facilitating the numerical determination of the spectral function.

References: [1] M. Laine, O. Philipsen, P. Romatschke and M. Tassler, "Real-time static potential in hot QCD", JHEP **03** (2007) 054; [2] M. Laine, "A resummed perturbative estimate for the quarkonium spectral function in hot QCD", JHEP **05** (2007) 028; [3] M. Laine, O. Philipsen and M. Tassler,"Thermal imaginary part of a real-time static potential from classical lattice gauge theory simulations", arXiv:0707.2458; [4] D.Y. Grigoriev and V.A. Rubakov, "Soliton pair creation at finite temperatures. Numerical study in (1+1)-dimensions", Nucl. Phys. B **299** (1988) 67; [5] J.B. Kogut and L. Susskind, "Hamiltonian formulation of Wilson's lattice gauge theories", Phys. Rev. D **11** (1975) 395; [6] J. Ambjørn, T. Askgaard, H. Porter and M.E. Shaposhnikov, "Lattice simulations of electroweak sphaleron transitions in real time", Phys. Lett. B **244** (1990) 479; [7] E. Braaten and R.D. Pisarski, "Soft amplitudes in hot gauge theories: a general analysis", Nucl. Phys. B **337** (1990) 569; J.C. Taylor and S.M.H. Wong, "The effective action of Hard Thermal Loops in QCD", Nucl. Phys. B **346** (1990) 115; [8] J.P. Blaizot and E. Iancu, "Kinetic equations for long wavelength excitations of the quark - gluon plasma", Phys. Rev. Lett. **70** (1993) 3376; [9] D. Bödeker, L.D. McLerran and A. Smilga, "Really computing nonperturbative real time correlation functions", Phys. Rev. D **52** (1995) 4675; [10] D. Bödeker, G.D. Moore and K. Rummukainen, "Chern-Simons number diffusion and hard thermal loops on the lattice", Phys. Rev. D **61** (2000) 056003; [11] A. Rebhan, P. Romatschke and M. Strickland, "Dynamics of quark-gluon plasma instabilities in discretized hard-loop approximation", JHEP **09** (2005) 041.